Finite element perturbation analysis of non-linear heat conduction problems with random field parameters

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Nomenclature

a	=	correlation length [m]
$\mathbf{C}$	=	finite element capacitance matrix
$\mathbf{C}^j$	=	local capacitance matrix corresponding to element j
E	=	mean value operator
$f(p)$	=	probability density function of p
$\mathbf{f}$	=	finite element thermal load vector
$\mathbf{f}^j$	=	local thermal load vector corresponding to element j
h	=	surface heat transfer coefficient [W/m^2°C]
$\mathbf{h}$	=	discretized surface heat transfer coefficient field vector
k	=	$k^0 g_k(T)$ thermal conductivity [W/m°C]
k^0	=	thermal conductivity at reference temperature [W/m°C]
$\mathbf{k}^0$	=	discretized thermal conductivity field vector
$\mathbf{K}$	=	finite element stiffness matrix
$\mathbf{K}^j$	=	local stiffness matrix corresponding to element j
L	=	length of slab [m]
n_{BC}	=	dimension of $\mathbf{h}$ and $\mathbf{T}_\infty$
n_Q	=	dimension of $\mathbf{Q}$
n_{el}	=	number of finite elements
n_{nod}	=	number of nodes
n_{TP}	=	dimension of $\mathbf{k}^0$, $\rho\mathbf{c}^0$ and $\mathbf{Q}$
p	=	arbitrary parameter
Q	=	heat generation rate [W/m^3]
$\mathbf{Q}$	=	discretized heat generation field vector
S	=	boundary surface
t	=	time [s]
T	=	temperature [°C] or [K]
T_0	=	initial temperature [°C]
$\mathbf{T}_0$	=	discretized initial temperature field vector
T_∞	=	ambient temperature [°C]
$\mathbf{T}_\infty$	=	discretized ambient temperature field vector
$\mathbf{u}$	=	nodal temperature vector
V	=	volume
$\mathbf{V}_{\mathbf{a,b}}$	=	covariance matrix of random vectors $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{x}$	=	position vector
x,y	=	spatial co-ordinates
β	=	temperature coefficient of k and ρc
Δ_t	=	time step [s]
λ	=	eigenvalue
ϕ^j	=	shape function vector corresponding to element j
ψ	=	matrix of eigenvectors
ρc	=	$\rho c^0 g_{\rho c}$: volumetric heat capacity [J/m^3°C]
ρc^0	=	volumetric heat capacity at reference temperature [J/m^3°C]
$\rho\mathbf{c}^0$	=	discretized volumetric heat capacity field vector
σ	=	standard deviation

Note: The symbols defined above are subject to alteration on occasion

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Introduction

The numerical analysis of conduction heat transfer by various methods such as the finite element and finite difference method is well established. For this purpose, it is usually assumed that the parameters relevant to the heat transfer process are accurately known. In reality many of these parameters may vary in a random way as a function of the spatial co-ordinates and can only be described appropriately by means of random fields. As a consequence, the temperature in the conductive medium at arbitrary space-time co-ordinates is also random and can only be specified meaningfully by means of statistical characteristics such as its mean value, variance and probability density function.

A straightforward statistical approach to the solution of heat conduction problems with random field parameters is the Monte Carlo method. In this method a sample of the random parameters is generated on the computer and the corresponding heat transfer problem is numerically solved. This procedure is repeated several times and, finally, the mean values and variances but also higher order moments can be estimated using common statistical techniques, and tests of hypothesis and significance can be performed. While the Monte Carlo method yields a rather complete picture of the stochastic properties of the temperature field, the large number of runs (typically > 1,000) necessary to obtain results with an acceptable accuracy and the considerable amount of corresponding computer time limits its applicability.

Alternatively, a probabilistic perturbation method may be used. This method is based on the computation of the propagation of an infinitesimal perturbation of the (stochastic) parameters during the process. Mean values and (co)variances of the process variables can then easily be evaluated. This method is used widely for stochastic elliptic and hyperbolic problems in structural analysis[1-4]. Nicolaï and De Baerdemaeker[5,6] applied the perturbation method to parabolic problems with random variable thermophysical parameters. Sluzalec[7] applied the perturbation algorithm outlined in Liu *et al.*[4] to linear parabolic problems with random field parameters. An essentially identical algorithm was applied by Fadale and Emery[8] to some test cases. In both cases the algorithm is very general and does not cope with non-linearities, random initial conditions and possible correlations between the random parameters. In this article the algorithm of Sluzalec[7] and Fadale and Emery[8] will be modified and extended to include these features.

Finite element analysis of non-linear heat conduction

It is assumed that the heat transfer is governed by the Fourier equation

$$\nabla k \nabla T + Q = \rho c \frac{\partial T}{\partial t} \tag{1}$$

where

- T : temperature [°C]
- t : time[s]

- k : thermal conductivity (W/m°C]
- ρc : volumetric heat capacity [J/m^3°C]
- Q : heat generation per unit volume [W/m^3]

Equation (1) is defined over some spatial domain V. The thermophysical properties k and ρc may be a function of the temperature and can be written as:

$$k = k^0 g_k(T) \tag{2}$$

$$\rho c = \rho c^0 g_{\rho c}(T) \tag{3}$$

where k^0 and ρc^0 are the values of k and ρc, respectively, at a known reference temperature T^0; the functions $g_k(T)$ and $g_{\rho c}(T)$ describe the temperature dependence of k and ρc, respectively, and are equal to unity at the reference temperature.

The initial condition at each position $\mathbf{x}$ is defined as

$$T(\mathbf{x}, t) = T_0(\mathbf{x}) \qquad \text{at} \quad t = t_0 \tag{4}$$

where $T_0(\mathbf{x})$ is a known function. It is further assumed that at the boundary surface S of the heated object convection boundary conditions may occur

$$k\frac{\partial}{\partial n}T = h[T_\infty - T] \qquad \text{at } S \tag{5}$$

here

- T_∞ : ambient temperature [°C]
- h : surface heat transfer coefficient [W/m^2°C]
- n : outward normal to the surface

Non-linear boundary conditions such as radiation are not considered. Further, the surface heat transfer coefficient is assumed to be independent of the temperature.

Equation (1) subject to (4) and (5) can be solved conveniently by means of the finite element method. For this purpose the continuum is sub-divided in elements of variable size and shape which are interconnected in a finite number n_{nod} of nodal points. In every element the unknown temperature is approximated by a low order interpolating polynomial

$$u^j(t) = \boldsymbol{\phi}^{jT}\mathbf{u}^j(t) \tag{6}$$

where $u^j(t)$ is the approximate temperature in element j, $\mathbf{u}^j(t)$ is the vector containing the nodal temperatures in element j, and ϕ^j is the vector of shape functions corresponding to element j. Applying the Galerkin method to equation (1) subject to (4) and (5) then yields the following non-linear differential system[9]

$$\mathbf{C}(\mathbf{u})\frac{d}{dt}\mathbf{u} + \mathbf{K}(\mathbf{u})\mathbf{u} = \mathbf{f} \tag{7}$$

with $\mathbf{u} = [u_1\ u_2 \ldots u_{n_{\text{nod}}}]^T$ the overall nodal temperature vector, $\mathbf{C}$ the capacitance matrix and $\mathbf{K}$ the stiffness matrix, both $n_{\text{nod}} \times n_{\text{nod}}$ matrices, and $\mathbf{f}$ a $n_{\text{nod}} \times 1$ vector. The system (7) can be solved by means of the finite difference method. For the construction of the global finite element matrices $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{f}$, it is most convenient from the programming point of view to first assemble the contributions of each element (the "element matrices" $\mathbf{C}^j$, $\mathbf{K}^j$ and $\mathbf{f}^j$)

$$\mathbf{C}^j = \int_{V^j} \rho c \boldsymbol{\phi}^j \boldsymbol{\phi}^{jT} dV \tag{8}$$

$$\mathbf{K}^j = \int_{V^j} k\mathbf{B}^j\mathbf{B}^{jT} dV + \int_{S^j} h\boldsymbol{\phi}^j\boldsymbol{\phi}^{jT} dS \tag{9}$$

$$\mathbf{f}^j = \int_{S^j} hT_\infty \boldsymbol{\phi}^j dS + \int_{V^j} Q\boldsymbol{\phi}^j dV \tag{10}$$

with

$$\mathbf{B}^j = \frac{\partial \boldsymbol{\phi}^j}{\partial \mathbf{x}}$$

and n_{el} the number of finite elements. The element matrices are then incorporated in the global matrices.

Random field parameters

Parameters which can vary in a random way as a function of the spatial co-ordinates are described most appropriately by means of random fields. It has been assumed further that the random fields are homogeneous, which means that their probabilistic characteristics do not change as a function of the spatial co-ordinates. A homogeneous random field $p(\mathbf{x})$ is fully characterized by means of its probability density function $f(p, \mathbf{x})$. In reality often only the mean value $\bar{p}(\mathbf{x})$ and its covariance function $\boldsymbol{V}_{p,p}(\Delta\mathbf{x})$ are known

$$\bar{p}(\mathbf{x}) = \bar{p} = \mathrm{E}(p) = \int_{-\infty}^{\infty} pf(p, \mathbf{x})dp$$

$$V_{p,p}(\Delta\mathbf{x}) = \mathrm{E}\{[p(\mathbf{x}) - \bar{p}][p(\mathbf{x} + \Delta\mathbf{x}) - \bar{p}]\} \tag{11}$$

$$= \int_{-\infty}^{\infty} [p(\mathbf{x}) - \bar{p}][p(\mathbf{x} + \Delta\mathbf{x}) - \bar{p}] f(p, \mathbf{x})dp \tag{12}$$

where E denotes the mean value operator. The (univariate) random field concept can easily be extended to bivariate random fields.

For further analysis the following four classes of random heat transfer parameters are distinguished:

(1) thermophysical properties k and ρc;

(2) process parameters h and T_∞;

(3) heat generation rate Q;

(4) initial condition T_0.

The classes are assumed to be mutually uncorrelated. However, quantities inside a class may be correlated. These assumptions are based on physical considerations. For example, k and ρc may be correlated since they both depend on the chemical composition and physical structure of the conductive medium. On the other hand, there is no a priori reason why the thermophysical properties and the boundary condition parameters h and T_∞ should be correlated.

If k and ρc are temperature dependent according to equation (2) and (3), it is assumed that k^0 and ρc^0 are random, while $g_k(T)$ and $g_{\rho c}(T)$ are deterministic functions.

It is assumed further that all quantities are of the random field type, with the random variable problem as a special subcase. Q and T_0 are homogeneous univariate Gaussian random fields with means $\overline{Q}$ and $\overline{T}_0$, and covariance functions $V_Q(\Delta\mathbf{x})$ and $V_{T0}(\Delta\mathbf{x})$, respectively. $[k^0\ \rho c^0]^T$ and $[h\ T_\infty]^T$ are homogeneous bivariate Gaussian random fields with mean $[\overline{k^0}\ \overline{\rho c^0}]^T$ and $[\overline{h}\ \overline{T}_\infty]^T$, and covariance function $V_{TP}(\Delta\mathbf{x})$ and $V_{BC}(\Delta\mathbf{x})$, respectively, where TP and BC stand for "thermophysical properties" and "boundary condition". It is assumed that the probability density functions corresponding to parameters which for physical reasons can only take on values of a limited range (thermophysical properties, surface heat transfer coefficient, ambient temperature and initial temperature) are truncated properly.

For computational purposes the random fields are spatially discretized resulting in random vectors of appropriate dimension. Several discretization methods have been suggested in the literature[4,10,11]. In this article k^0, ρc^0, and Q are discretized in the midpoints of the finite elements; h and T_∞ in the midpoints of the element edges which are exposed to a convection boundary condition; and T_0 in the nodes of the finite element grid. The dimensions of the corresponding random vectors are tabulated in Table I.

Parameter	Random vector	Dimension
k^0	$\mathbf{k}^0$	n_{TP}
ρc^0	$\rho\mathbf{c}^0$	n_{TP}
h	$\mathbf{h}$	n_{BC}
T_∞	$\mathbf{T}_\infty$	n_{BC}
Q	$\mathbf{Q}$	n_Q
T_0	$\mathbf{T}_0$	n_{nod}

Table I.
Discretization of random field parameters

The (cross) covariance matrix of two random vectors is denoted as

$$\mathrm{E}[(\mathbf{k}^0 - \overline{\mathbf{k}^0})(\boldsymbol{\rho}\mathbf{c}^0 - \overline{\boldsymbol{\rho}\mathbf{c}^0})^T] = \mathbf{V}_{\mathbf{k}^0,\boldsymbol{\rho}\mathbf{c}^0}$$

The entry of $\mathbf{V}_{\mathbf{k}^0,\boldsymbol{\rho}\mathbf{c}^0}$ with row index i and column index j, which represents the covariance between entry with index i of the first vector and entry with index j of the second vector, is denoted by $V_{k_i^0,\,\rho c_j^0}$.

Perturbation algorithm

The starting point of the analysis is the finite element formulation of the Fourier equation

$$\mathbf{C}(\boldsymbol{\rho}\mathbf{c}^0, \mathbf{u})\,\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{u}(\mathbf{k}^0, \boldsymbol{\rho}\mathbf{c}^0, \mathbf{h}, \mathbf{T}_\infty, \mathbf{Q}, \mathbf{T}_0) + \mathbf{K}(\mathbf{k}^0, \mathbf{h}, \mathbf{u})\,\mathbf{u}(\mathbf{k}^0, \boldsymbol{\rho}\mathbf{c}^0, \mathbf{h}, \mathbf{T}_\infty, \mathbf{Q}, \mathbf{T}_0) = \mathbf{f}(\mathbf{h}, \mathbf{T}_\infty, \mathbf{Q}) \tag{13}$$

in which the dependency of the system matrices and vectors on the (spatially discretized) random parameters is shown. The matrices and vectors in the system (13) can be expanded into a first order Taylor series:

$$\mathbf{C} \cong \overline{\mathbf{C}} + \sum_{i=1}^{n_{TP}} \frac{\partial \mathbf{C}}{\partial \rho c_i^0}\Delta\rho c_i^0 + \sum_{j=1}^{n_{nod}} \frac{\partial \mathbf{C}}{\partial u_j}\left(\frac{\partial u_j}{\partial \boldsymbol{\rho}\mathbf{c}^0}\Delta\boldsymbol{\rho}\mathbf{c}^0 + \frac{\partial u_j}{\partial \mathbf{k}^0}\Delta\mathbf{k}^0 + \frac{\partial u_j}{\partial \mathbf{h}}\Delta\mathbf{h} + \frac{\partial u_j}{\partial \mathbf{T}_\infty}\Delta\mathbf{T}_\infty + \frac{\partial u_j}{\partial \mathbf{Q}}\Delta\mathbf{Q} + \frac{\partial u_j}{\partial \mathbf{T}_0}\Delta\mathbf{T}_0\right) \tag{14}$$

$$\mathbf{K} \cong \overline{\mathbf{K}} + \sum_{i=1}^{n_{TP}} \frac{\partial \mathbf{K}}{\partial k_i^0}\Delta k_i^0 + \sum_{j=1}^{n_{nod}} \frac{\partial \mathbf{K}}{\partial u_j}\left(\frac{\partial u_j}{\partial \boldsymbol{\rho}\mathbf{c}^0}\Delta\boldsymbol{\rho}\mathbf{c}^0 + \frac{\partial u_j}{\partial \mathbf{k}^0}\Delta\mathbf{k}^0 + \frac{\partial u_j}{\partial \mathbf{h}}\Delta\mathbf{h} + \frac{\partial u_j}{\partial \mathbf{T}_\infty}\Delta\mathbf{T}_\infty + \frac{\partial u_j}{\partial \mathbf{Q}}\Delta\mathbf{Q} + \frac{\partial u_j}{\partial \mathbf{T}_0}\Delta\mathbf{T}_0\right) + \sum_{i=1}^{n_{BC}} \frac{\partial \mathbf{K}}{\partial h_i}\Delta h_i \tag{15}$$

$$\mathbf{u} \cong \overline{\mathbf{u}} + \frac{\partial \mathbf{u}}{\partial \mathbf{k}^0}\Delta\mathbf{k}^0 + \frac{\partial \mathbf{u}}{\partial \boldsymbol{\rho}\mathbf{c}^0}\Delta\boldsymbol{\rho}\mathbf{c}^0 + \frac{\partial \mathbf{u}}{\partial \mathbf{h}}\Delta\mathbf{h} + \frac{\partial \mathbf{u}}{\partial \mathbf{T}_\infty}\Delta\mathbf{T}_\infty + \frac{\partial \mathbf{u}}{\partial \mathbf{Q}}\Delta\mathbf{Q} + \frac{\partial \mathbf{u}}{\partial \mathbf{T}_0}\Delta\mathbf{T}_0 \tag{16}$$

$$\mathbf{f} \cong \overline{\mathbf{f}} + \frac{\partial \mathbf{f}}{\partial \mathbf{h}}\Delta\mathbf{h} + \frac{\partial \mathbf{f}}{\partial \mathbf{T}_\infty}\Delta\mathbf{T}_\infty + \frac{\partial \mathbf{f}}{\partial \mathbf{Q}}\Delta\mathbf{Q} \tag{17}$$

where

$$\overline{\mathbf{C}} \triangleq \mathbf{C}(\overline{\boldsymbol{\rho}\mathbf{c}^0}, \overline{\mathbf{u}}) \tag{18}$$

$$\bar{\mathbf{K}} \triangleq \mathbf{K}(\bar{\mathbf{k}}^0, \bar{\mathbf{h}}, \bar{\mathbf{u}}) \tag{19}$$

$$\bar{\mathbf{f}} \triangleq \mathbf{f}(\bar{\mathbf{h}}, \bar{\mathbf{T}}_\infty, \bar{\mathbf{Q}}) \tag{20}$$

and $\bar{\mathbf{u}}$ is the solution of the deterministic (non-linear) heat transfer problem

$$\bar{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\bar{\mathbf{u}} + \bar{\mathbf{K}}\bar{\mathbf{u}} = \bar{\mathbf{f}} \tag{21}$$

with the initial condition

$$\bar{\mathbf{u}}(t = 0) = \bar{\mathbf{u}}_0 \tag{22}$$

Note that an expression like $\frac{\partial \mathbf{C}}{\partial \rho c_i^0}$ denotes the partial derivative of $\mathbf{C}$ with respect to the ith component of the random vector $\rho\mathbf{c}^0$.

The derivatives of the matrices in the Taylor expansion are to be evaluated using the mean parameters, e.g.

$$\frac{\partial \mathbf{f}}{\partial \mathbf{T}_\infty} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{T}_\infty}\right|_{\bar{\mathbf{T}}_\infty, \bar{\mathbf{h}}}$$

After substitution of (14)-(17) in equation (13) and combining corresponding terms, the following system is obtained

$$\bar{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathbf{u}}{\partial \rho c_i^0}\right) + \bar{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial \rho c_i^0} = -\frac{\partial \mathbf{C}}{\partial \rho c_i^0}\frac{\mathrm{d}}{\mathrm{dt}}\bar{\mathbf{u}} \qquad i = 1, \ldots, n_{TP} \tag{23}$$

$$\bar{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathbf{u}}{\partial k_i^0}\right) + \bar{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial k_i^0} = -\frac{\partial \mathbf{K}}{\partial k_i^0}\bar{\mathbf{u}} \qquad i = 1, \ldots, n_{TP} \tag{24}$$

$$\bar{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathbf{u}}{\partial h_i}\right) + \bar{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial h_i} = \frac{\partial \mathbf{f}}{\partial h_i} - \frac{\partial \mathbf{K}}{\partial h_i}\bar{\mathbf{u}} \qquad i = 1, \ldots, n_{BC} \tag{25}$$

$$\bar{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathbf{u}}{\partial T_{\infty_i}}\right) + \bar{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial T_{\infty_i}} = \frac{\partial \mathbf{f}}{\partial T_{\infty_i}} \qquad i = 1, \ldots, n_{BC} \tag{26}$$

$$\bar{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathbf{u}}{\partial Q_i}\right) + \bar{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial Q_i} = \frac{\partial \mathbf{f}}{\partial Q_i} \qquad i = 1, \ldots, n_Q \tag{27}$$

$$\bar{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathbf{u}}{\partial T_{0i}}\right) + \bar{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial T_{0i}} = \mathbf{0} \qquad i = 1, \ldots, n_{nod} \tag{28}$$

with

$$\overset{\triangle}{\mathbf{K}} = \bar{\mathbf{K}} + \left[\, \frac{\partial \mathbf{K}}{\partial u_1}\bar{\mathbf{u}} \;\middle|\; \ldots \;\middle|\; \frac{\partial \mathbf{K}}{\partial u_{n_{nod}}}\bar{\mathbf{u}} \,\right] \tag{29}$$

$$\overset{\nabla}{\mathbf{K}} = \overset{\Delta}{\mathbf{K}} + \left[\frac{\partial \mathbf{C}}{\partial u_1}\frac{\mathrm{d}}{\mathrm{dt}}\bar{\mathbf{u}} \;\middle|\; \cdots \;\middle|\; \frac{\partial \mathbf{C}}{\partial u_{n_{nod}}}\frac{\mathrm{d}}{\mathrm{dt}}\bar{\mathbf{u}} \right] \tag{30}$$

Obviously, for linear problems

$$\frac{\partial \mathbf{C}}{\partial u_i} = \mathbf{0}, \qquad i = 1, \ldots, n_{nod} \tag{31}$$

$$\frac{\partial \mathbf{K}}{\partial u_i} = \mathbf{0}, \qquad i = 1, \ldots, n_{nod} \tag{32}$$

so that

$$\overset{\nabla}{\mathbf{K}} = \overset{\Delta}{\mathbf{K}} = \overline{\mathbf{K}}$$

Applying the mean value operator on equation (16) yields

$$\mathrm{E}(\mathbf{u}) \cong \bar{\mathbf{u}}$$

This means that a first order approximation of the mean temperature vector can be found by solving the original (deterministic) heat transfer problem using the mean values of the parameters. The covariance matrix $\mathbf{V}_{\mathbf{u},\mathbf{u}}$ at an arbitrary time t can be computed from

$$\begin{aligned}
\mathbf{V}_{\mathbf{u},\mathbf{u}} &= \mathrm{E}[(\mathbf{u}-\bar{\mathbf{u}})(\mathbf{u}-\bar{\mathbf{u}})^T] \\
&= \frac{\partial \mathbf{u}}{\partial \boldsymbol{\rho}\mathbf{c}^0}\mathbf{V}_{\boldsymbol{\rho}\mathbf{c}^0,\boldsymbol{\rho}\mathbf{c}^0}\left(\frac{\partial \mathbf{u}}{\partial \boldsymbol{\rho}\mathbf{c}^0}\right)^T + \frac{\partial \mathbf{u}}{\partial \boldsymbol{\rho}\mathbf{c}^0}\mathbf{V}_{\boldsymbol{\rho}\mathbf{c}^0,\mathbf{k}^0}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{k}^0}\right)^T \\
&\quad + \frac{\partial \mathbf{u}}{\partial \mathbf{k}^0}\mathbf{V}_{\mathbf{k}^0,\boldsymbol{\rho}\mathbf{c}^0}\left(\frac{\partial \mathbf{u}}{\partial \boldsymbol{\rho}\mathbf{c}^0}\right)^T + \frac{\partial \mathbf{u}}{\partial \mathbf{k}^0}\mathbf{V}_{\mathbf{k}^0,\mathbf{k}^0}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{k}^0}\right)^T \\
&\quad + \frac{\partial \mathbf{u}}{\partial \mathbf{h}}\mathbf{V}_{\mathbf{h},\mathbf{h}}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{h}}\right)^T + \frac{\partial \mathbf{u}}{\partial \mathbf{h}}\mathbf{V}_{\mathbf{h},\mathbf{T}_\infty}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{T}_\infty}\right)^T \\
&\quad + \frac{\partial \mathbf{u}}{\partial \mathbf{T}_\infty}\mathbf{V}_{\mathbf{T}_\infty,\mathbf{h}}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{h}}\right)^T + \frac{\partial \mathbf{u}}{\partial \mathbf{T}_\infty}\mathbf{V}_{\mathbf{T}_\infty,\mathbf{T}_\infty}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{T}_\infty}\right)^T \\
&\quad + \frac{\partial \mathbf{u}}{\partial \mathbf{Q}}\mathbf{V}_{\mathbf{Q},\mathbf{Q}}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{Q}}\right)^T + \frac{\partial \mathbf{u}}{\partial \mathbf{T}_0}\mathbf{V}_{\mathbf{T}_0,\mathbf{T}_0}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{T}_0}\right)^T
\end{aligned} \tag{33}$$

At $t = 0$,

$$\frac{\partial \mathbf{u}}{\partial \rho c_i^0} = \frac{\partial \mathbf{u}}{\partial k_i^0} = \mathbf{0} \qquad i = 1, \ldots, n_{TP} \tag{34}$$

$$\frac{\partial \mathbf{u}}{\partial h_i} = \frac{\partial \mathbf{u}}{\partial T_{\infty_i}} = \mathbf{0} \qquad i = 1, \ldots, n_{BC} \tag{35}$$

$$\frac{\partial \mathbf{u}}{\partial Q_i} = \mathbf{0} \qquad i = 1, \ldots, n_Q \tag{36}$$

so that, after substitution of (34)-(36) in (33),

$$\mathbf{V}_{\mathbf{u},\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial \mathbf{T}_0} \mathbf{V}_{\mathbf{T}_0,\mathbf{T}_0} \frac{\partial \mathbf{u}}{\partial \mathbf{T}_0}^T \qquad \text{at } t = t_0 \tag{37}$$

Since

$$\mathbf{V}_{\mathbf{u},\mathbf{u}} \equiv \mathbf{V}_{\mathbf{T}_0,\mathbf{T}_0} \qquad \text{at } t = t_0,$$

equation (37) is satisfied if

$$\frac{\partial \mathbf{u}}{\partial \mathbf{T}_0} \equiv \mathbf{I} \qquad \text{at } t = t_0 \tag{38}$$

If only the variances $\sigma^2_{u_i}$, $i = 1, n_{\text{nod}}$, which are the diagonal entries of $\mathbf{V}_{\mathbf{u},\mathbf{u}}$, are needed, equation (33) reduces to

$$\begin{aligned}
\sigma^2_{u_i} = {} & \frac{\partial u_i}{\partial \boldsymbol{\rho}\mathbf{c}^0} \mathbf{V}_{\boldsymbol{\rho}\mathbf{c}^0,\boldsymbol{\rho}\mathbf{c}^0} \left(\frac{\partial u_i}{\partial \boldsymbol{\rho}\mathbf{c}^0}\right)^T + \frac{\partial u_i}{\partial \boldsymbol{\rho}\mathbf{c}^0} \mathbf{V}_{\boldsymbol{\rho}\mathbf{c}^0,\mathbf{k}^0} \left(\frac{\partial u_i}{\partial \mathbf{k}^0}\right)^T \\
& + \frac{\partial u_i}{\partial \mathbf{k}^0} \mathbf{V}_{\mathbf{k}^0,\boldsymbol{\rho}\mathbf{c}^0} \left(\frac{\partial u_i}{\partial \boldsymbol{\rho}\mathbf{c}^0}\right)^T + \frac{\partial u_i}{\partial \mathbf{k}^0} \mathbf{V}_{\mathbf{k}^0,\mathbf{k}^0} \left(\frac{\partial u_i}{\partial \mathbf{k}^0}\right)^T \\
& + \frac{\partial u_i}{\partial \mathbf{h}} \mathbf{V}_{\mathbf{h},\mathbf{h}} \left(\frac{\partial u_i}{\partial \mathbf{h}}\right)^T + \frac{\partial u_i}{\partial \mathbf{h}} \mathbf{V}_{\mathbf{h},\mathbf{T}_\infty} \left(\frac{\partial u_i}{\partial \mathbf{T}_\infty}\right)^T \\
& + \frac{\partial u_i}{\partial \mathbf{T}_\infty} \mathbf{V}_{\mathbf{T}_\infty,\mathbf{h}} \left(\frac{\partial u_i}{\partial \mathbf{h}}\right)^T + \frac{\partial u_i}{\partial \mathbf{T}_\infty} \mathbf{V}_{\mathbf{T}_\infty,\mathbf{T}_\infty} \left(\frac{\partial u_i}{\partial \mathbf{T}_\infty}\right)^T \\
& + \frac{\partial u_i}{\partial \mathbf{Q}} \mathbf{V}_{\mathbf{Q},\mathbf{Q}} \left(\frac{\partial u_i}{\partial \mathbf{Q}}\right)^T + \frac{\partial u_i}{\partial \mathbf{T}_0} \mathbf{V}_{\mathbf{T}_0,\mathbf{T}_0} \left(\frac{\partial u_i}{\partial \mathbf{T}_0}\right)^T
\end{aligned} \tag{39}$$

If the random fields are discretized by an appropriate discretization algorithm, the covariance matrices are positive definite so that the quadratic forms in (39) are all positive. This means that each uncertain parameter has a positive contribution to the temperature variance.

The global algorithm can then be summarized as follows.

Step 1. Compute $\bar{\mathbf{u}}$ from equation (21) with initial condition (22);

Step 2. Solve equations (23)-(28) for the sensitivity vectors $\partial \mathbf{u}/\partial \boldsymbol{\rho}\mathbf{c}^0_l$, $\partial \mathbf{u}/\partial \mathbf{k}^0_l$, $\partial \mathbf{u}/\partial \mathbf{h}_l$, … with corresponding initial conditions (34)-(36) and (38);

Step 3. Compute $\mathbf{V}_{\mathbf{uu}}(t)$ from equation (33) at the required time instances.

Equation (21) and equations (23)-(28) are linear and can be solved by means of finite differences. If the problem is linear then $\overline{\mathbf{K}} = \mathbf{K}$. As a consequence, if an implicit finite difference method is applied the computer time can be reduced. For example, in the implicit Euler method a matrix $\overline{\mathbf{K}} + \overline{\mathbf{C}}/\Delta t$ appears which must be triangularized. This triangularization must be accomplished only once, namely for the computation of $\bar{\mathbf{u}}$ according to equation (21). For non-linear problems $\overline{\mathbf{K}} \neq \mathbf{K}$ and the matrix $\mathbf{K} + \overline{\mathbf{C}}/\Delta t$ is time-varying. However, since

the latter matrix arises in the implicit Euler solution of equations (23)-(28) it must be triangularized only once each time step. Also, $\overset{\triangle}{\mathbf{K}}$ is, in general, non-symmetric. Consequently, the CPU time needed for the triangularization doubles in comparison to the linear case where $\overset{\triangle}{\mathbf{K}}$ is equal to $\overline{\mathbf{K}}$ and, hence, symmetric.

For steady state problems, the differential system (23)-(28) reduces to the following algebraic system

$$\begin{aligned}
\overset{\triangle}{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial k_i^0} &= -\frac{\partial \mathbf{K}}{\partial k_i^0}\bar{\mathbf{u}} && i = 1, \ldots, n_{TP}\\
\overset{\triangle}{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial h_i} &= \frac{\partial \mathbf{f}}{\partial h_i} - \frac{\partial \mathbf{K}}{\partial h_i}\bar{\mathbf{u}} && i = 1, \ldots, n_{BC}\\
\overset{\triangle}{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial T_{\infty_i}} &= \frac{\partial \mathbf{f}}{\partial T_{\infty_i}} && i = 1, \ldots, n_{BC}\\
\overset{\triangle}{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial Q_i} &= \frac{\partial \mathbf{f}}{\partial Q_i} && i = 1, \ldots, n_Q
\end{aligned}$$

where $\bar{\mathbf{u}}$ is the solution of the following steady state heat transfer problem

$$\overline{\mathbf{K}}\bar{\mathbf{u}} = \bar{\mathbf{f}} \tag{40}$$

The terms involving ρc and T_0 disappear correspondingly from equation (33).

A special case: random variable parameters

If a random quantity is of the random variable type, the algorithms presented above can be simplified. For example, for a transient linear heat transfer problem with random variable thermal conductivity, equations (23), (24) and (33) become

$$\begin{aligned}
&\overline{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\bar{\mathbf{u}} + \overline{\mathbf{K}}\bar{\mathbf{u}} = \bar{\mathbf{f}}\\
&\overline{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathbf{u}}{\partial \rho c}\right) + \overline{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial \rho c} = -\frac{\partial \mathbf{C}}{\partial \rho c}\frac{\mathrm{d}}{\mathrm{dt}}\bar{\mathbf{u}}\\
&\overline{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathbf{u}}{\partial k}\right) + \overline{\mathbf{K}}\frac{\partial \mathbf{u}}{\partial k} = -\frac{\partial \mathbf{K}}{\partial k}\bar{\mathbf{u}}
\end{aligned} \tag{41}$$

$$\begin{aligned}
\mathbf{V}_{\mathbf{u},\mathbf{u}} = {} & \frac{\partial \mathbf{u}}{\partial \rho c}\left(\frac{\partial \mathbf{u}}{\partial \rho c}\right)^T \sigma_{\rho c}^2 + \frac{\partial \mathbf{u}}{\partial \rho c}\left(\frac{\partial \mathbf{u}}{\partial k}\right)^T \sigma_{k,\rho c}\\
& + \frac{\partial \mathbf{u}}{\partial k}\left(\frac{\partial \mathbf{u}}{\partial \rho c}\right)^T \sigma_{k,\rho c} + \frac{\partial \mathbf{u}}{\partial k}\left(\frac{\partial \mathbf{u}}{\partial k}\right)^T \sigma_k^2
\end{aligned} \tag{42}$$

Derivatives of **C**, **K** *and* **f** *with respect to random parameters*

The derivatives of **C**, **K** and **f** with respect to the random parameters can be computed by differentiation of the element matrices and subsequent incorporation in the global derivative matrices. The following expressions are easily derived from equations (8)-(10), using (2) and (3):

$$\frac{\partial \mathbf{C}^j}{\partial \rho c_i^0} = \int_{Vj} f_{\rho c} \boldsymbol{\phi}^j \boldsymbol{\phi}^{jT} dV \qquad i = j$$
$$= \mathbf{0} \qquad i \neq j$$

$$\frac{\partial \mathbf{K}^j}{\partial k_i^0} = \int_{Vj} f_k \mathbf{B}^j \mathbf{B}^{jT} dV \qquad i = j$$
$$= \mathbf{0} \qquad i \neq j$$

$$\frac{\partial \mathbf{K}^j}{\partial h_i} = \int_{Sj} \boldsymbol{\phi}^j \boldsymbol{\phi}^{jT} dS \qquad i = j$$
$$= \mathbf{0} \qquad i \neq j$$

$$\frac{\partial \mathbf{f}^j}{\partial h_i} = \int_{Sj} T_\infty \boldsymbol{\phi}^j dS \qquad i = j$$
$$= \mathbf{0} \qquad i \neq j$$

$$\frac{\partial \mathbf{f}^j}{\partial T_{\infty_i}} = \int_{Sj} h \boldsymbol{\phi}^j dS \qquad i = j$$
$$= \mathbf{0} \qquad i \neq j$$

$$\frac{\partial \mathbf{f}^j}{\partial Q_i} = \int_{Vj} \boldsymbol{\phi}^j dV \qquad i = j$$
$$= \mathbf{0} \qquad i \neq j$$

These particularly simple expressions are a consequence of the midpoint discretization method. Clearly the derivative matrices are very sparse: only the entries corresponding to element *j* are non-zero. This result can be exploited in

the computation of the matrix multiplications in the right-hand side of the perturbation system and variance propagation equations.

The derivatives of **K** and **C** with respect to the temperature vector **u** can be assembled using the following expressions:

$$\frac{\partial \mathbf{K}^j}{\partial u_i} = \int_{Vj} \mathbf{B}^j \mathbf{B}^{jT} (dk/dT) \phi_i^j dV \quad \text{node } i \in \text{element } j$$

$$= \mathbf{0} \quad \text{else}$$

$$\frac{\partial \mathbf{C}^j}{\partial u_i} = \int_{Vj} \boldsymbol{\phi}^j \boldsymbol{\phi}^{jT} (d\rho c/dT) \phi_i^j dV \quad \text{node } i \in \text{element } j$$

$$= \mathbf{0} \quad \text{else}$$

Reduction of the dimension of the discretized random fields

The CPU time required in the perturbation algorithm for the computation of the covariance matrix according to equation (33) is proportional to the square of the dimensions of the random vectors resulting from the discretization of the random field parameters. If these dimensions are large, the total CPU time will be determined by the time spent in evaluating these equations. It therefore seems to be worthwhile to reduce the dimension of these random vectors as suggested by Liu *et al.*[4].

Consider a transient linear heat transfer problem with random field thermophysical parameters k and ρc. Assume that k and ρc are discretized and that the covariance function of **k** and $\rho \mathbf{c}$ is given by

$$\mathbf{V}_{TP}(\Delta \mathbf{x}) = \begin{bmatrix} \sigma_{k^0}^2 & \sigma_{k^0 . \rho c^0} \\ \sigma_{k^0 . \rho c^0} & \sigma_{\rho c^0}^2 \end{bmatrix} R_{TP}(\Delta \mathbf{x}) \tag{43}$$

with $R_{TP}(\Delta \mathbf{x})$ the correlation function which is equal to unity for $\Delta \mathbf{x} = \mathbf{0}$. A transformation

$$\tilde{\mathbf{k}} = \boldsymbol{\Psi} \mathbf{k}$$

$$\tilde{\rho \mathbf{c}} = \boldsymbol{\Psi} \rho \mathbf{c}$$

is defined such that the correlation matrix $\tilde{\mathbf{R}}_{TP}$ of the transformed variables is a diagonal matrix and that ψ is orthogonal

$$\boldsymbol{\Psi}^T \boldsymbol{\Psi} = \boldsymbol{\Psi} \boldsymbol{\Psi}^T = \mathbf{I}$$

It is easy to show that this involves the solution of the eigenproblem

$$\boldsymbol{\Psi} \tilde{\mathbf{R}}_{TP} = \mathbf{R}_{TP} \boldsymbol{\Psi} \tag{44}$$

where $\tilde{\mathbf{R}}_{TP}$ is the diagonal matrix containing the eigenvalues, and ψ is the matrix of normalized eigenvectors. Note that the eigendecomposition (44) exists if $\mathbf{R}_{TP}$ is positive-definite. This is the case if a suitable spatial discretization procedure is applied[12]. The (co)variances of the transformed variables $\tilde{k}_i$ and $\tilde{\rho c}_i$ are equal to

$$\begin{aligned}
\sigma^2_{\tilde{k}_i} &= \sigma^2_k \lambda_i \\
\sigma^2_{\tilde{\rho c}_i} &= \sigma^2_{\rho c} \lambda_i \\
\sigma_{\tilde{\rho c}_i, \tilde{k}_j} &= \sigma_{k,\rho c} \lambda_i, \qquad i = j \\
&= 0, \qquad i \neq j
\end{aligned}$$

where λ_i, $i = 1, \ldots, n_{TP}$ are the eigenvalues of $\mathbf{R}_{TP}$, and, hence, the diagonal entries of $\tilde{\mathbf{R}}_{TP}$. Further, from (44) it follows that

$$\mathbf{R}_{TP} = \boldsymbol{\Psi} \tilde{\mathbf{R}}_{TP} \boldsymbol{\Psi}^T = \sum_{i=1}^{n_{TP}} \boldsymbol{\Psi}_i \lambda_i \boldsymbol{\Psi}_i^T \tag{45}$$

with ψ_i the ith eigenvector of $\mathbf{R}_{TP}$. Now, suppose that the eigenvalues λ_i are ordered from large to small, it can be expected that a reasonable approximation of $\mathbf{R}_{TP}$ can be obtained by evaluating only the first few, say $N < n_{TP}$, terms in the sum in equation (45). The perturbation algorithm is now applied using the transformed variables. This yields the following system

$$\begin{aligned}
\bar{\mathbf{C}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathbf{u}}{\partial \tilde{\rho c}_i} \right) + \mathbf{K} \frac{\partial \mathbf{u}}{\partial \tilde{\rho c}_i} &= -\frac{\partial \mathbf{C}}{\partial \tilde{\rho c}_i} \frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathbf{u}} \qquad i = 1, \ldots, n_{TP} \\
\bar{\mathbf{C}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathbf{u}}{\partial \tilde{k}_i} \right) + \mathbf{K} \frac{\partial \mathbf{u}}{\partial \tilde{k}_i} &= -\frac{\partial \mathbf{K}}{\partial \tilde{k}_i} \bar{\mathbf{u}} \qquad i = 1, \ldots, n_{TP}
\end{aligned}$$

The Jacobian matrices with respect to the transformed variables are a linear combination of those of the original variables. For example, from (44) it follows that

$$k_j = \sum_i \Psi_{i,j} \tilde{k}_i$$

Applying the chain rule yields

$$\begin{aligned}
\frac{\partial \mathbf{K}}{\partial \tilde{k}_i} &= \sum_j \frac{\partial \mathbf{K}}{\partial k_j} \frac{\partial k_j}{\partial \tilde{k}_i} \\
&= \sum_j \frac{\partial \mathbf{K}}{\partial k_j} \Psi_{i,j}
\end{aligned}$$

The covariance matrix is then computed from

$$\mathbf{V}_{\mathbf{u},\mathbf{u}} = \sigma_{\rho c}^2 \sum_{i=1}^{N} \lambda_i \frac{\partial \mathbf{u}}{\partial \tilde{\rho c}_i} \left(\frac{\partial \mathbf{u}}{\partial \tilde{\rho c}_i} \right)^T + \sigma_{k,\rho c} \sum_{i=1}^{N} \lambda_i \frac{\partial \mathbf{u}}{\partial \tilde{\rho c}_i} \left(\frac{\partial \mathbf{u}}{\partial \tilde{k}_i} \right)^T + \sigma_{k,\rho c} \sum_{i=1}^{N} \lambda_i \frac{\partial \mathbf{u}}{\partial \tilde{k}_i} \left(\frac{\partial \mathbf{u}}{\partial \tilde{\rho c}_i} \right)^T + \sigma_k^2 \sum_{i=1}^{N} \lambda_i \frac{\partial \mathbf{u}}{\partial \tilde{k}_i} \left(\frac{\partial \mathbf{u}}{\partial \tilde{k}_i} \right)^T$$

Numerical examples

In order to compare the first order perturbation method with the Monte Carlo in terms of accuracy and required computer time, some test problems were investigated. In all cases it has been assumed that all (univariate) random fields p are homogeneous with mean $\bar{p}$ and exponential covariance function

$$V_{p,p}(\Delta \mathbf{x}) = \sigma_p^2 R(\Delta \mathbf{x}) \tag{46}$$

$$R(\Delta \mathbf{x}) = \exp(-|\Delta \mathbf{x}|/a) \tag{47}$$

with σ_p^2 the variance of the field. The parameter a is called the correlation length and gives an indication of the smoothness of the random field. The bivariate homogeneous random fields $\mathbf{p} = [\overline{p}_1\ \overline{p}_2]^T$ are also homogeneous with mean $\bar{\mathbf{p}} = [\overline{p}_1\ \overline{p}_2]^T$ and covariance function

$$\mathbf{V}_{\mathbf{p},\mathbf{p}}(\Delta \mathbf{x}) = \begin{bmatrix} \sigma_{p_1}^2 & \sigma_{p_1,p_2} \\ \sigma_{p_1,p_2} & \sigma_{p_2}^2 \end{bmatrix} R(\Delta \mathbf{x}) \tag{48}$$

where $R(\Delta \mathbf{x})$ is defined by equation (47).

An existing finite element code for non-linear heat conduction analysis Dot[13] was modified. The element midpoints, required for the discretization of the random fields, were computed by Gauss-Legendre quadrature.

In the program, the eigendecomposition of the correlation matrices as described above is computed using Nag routine F02ABF.

For non-linear problems, $\overset{\nabla}{\mathbf{K}}$ is not equal to $\overline{\mathbf{K}}$ and unsymmetric in general. It has, however, the same sparse skyline structure as $\overline{\mathbf{K}}$. Therefore only the non-zero entries are kept in a vector and two additional (integer) vectors contain the row and column index of each entry. Each time step $\overset{\nabla}{\mathbf{K}}$ must be triangularized, but within each time step this must be done only once. The triangularization and the back-substitution are carried out by Nag routines F01BRF and F01BSF.

For the Monte Carlo analysis, the random fields were discretized as described above. Samples of the resulting random vectors were generated by means of Nag routines G05EAF and G05EZF.

All algorithms were programmed and executed on a HP-720 workstation.

Example 1. Parallel sided slab with heat generation

The first test problem consisted of a parallel sided metal slab with length L (L = 10cm) with uniform internal heat generation Q (10^6W/m^3) and random field thermophysical properties with the following characteristics: $\bar{k}$ =100 W/m°C; ρc =7·10^6 J/m^3°C; $\sigma_{k^0}/\bar{k}^0 = \sigma_{\rho c}{}^0/\bar{\rho c}^0 = 0.1$; $\sigma_{k^0,\,\rho c}{}^0/\sigma_{k^0}\sigma_{\rho c^0} = 0.5$; a = 5 cm. Further, $g_k(T) = g_{\rho c}(T) \equiv 1$. The slab is initially at a temperature T_0 = 0°C. The boundary at $x = +L$ is kept at a fixed temperature T_∞ (0°C) by setting h to an arbitrary large value. The boundary at x = 0cm is adiabatic.

For the finite element analysis the region $[0,L]$ was subdivided in ten identical quadrilateral elements with four nodes per element (only two-dimensional planar and axisymmetric elements are provided in the DOT finite element code). Δt was set at 10s for the transient analysis. Preliminary deterministic simulations showed that the global error of the finite element approximation with respect to the analytical solution was of the order 0.1°C.

In Figure 1 the mean temperature is shown for three different positions in the slab. It is clear that no visual distinction can be made between the mean temperatures computed by means of the different methods. The temperature variance is shown in Figure 2. It can be observed that the results of the (first order) perturbation analysis agree very well with those of both Monte Carlo analyses. The CPU time required for the perturbation algorithm (3s) was considerably below that required for the Monte Carlo analysis (24.4s for 100 simulation runs and 242s for 1,000 runs).

In Figure 3 the influence of the number of considered eigenvalues on the accuracy of the calculated variance at x = 0cm and t = 2,000s is investigated for different values of a/L. For this purpose the calculations were done using a grid of 50 elements. In all cases, only a few eigenvalues are required. This has

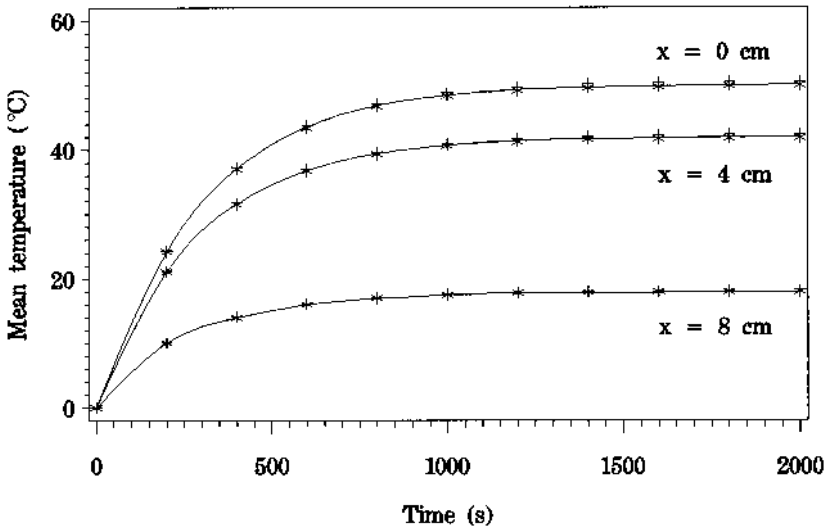

Figure 1. Mean transient temperature in a slab with random field thermophysical properties – first order perturbation; *: Monte Carlo (n_{MC} = 100); +: Monte Carlo (n_{MC} = 1,000)

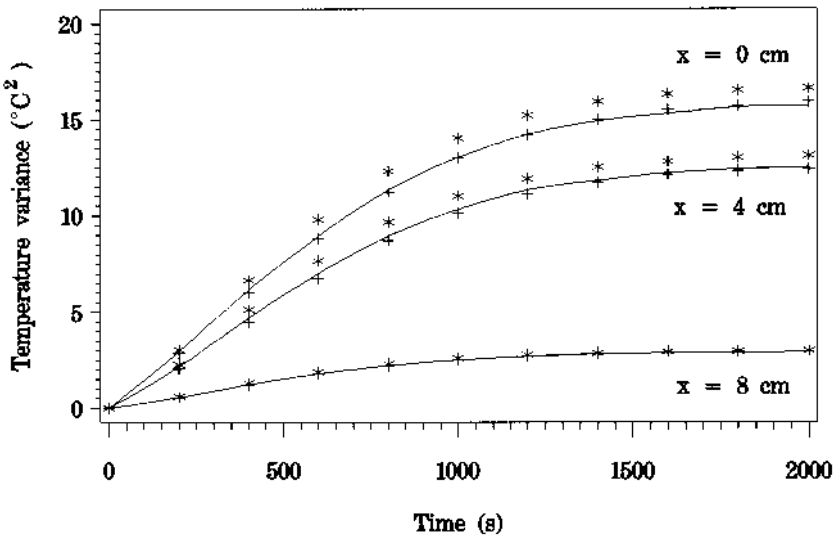

Figure 2.
Variance of the transient temperature in a slab with random field thermophysical properties – first order perturbation; *: Monte Carlo (n_{MC} = 100); +: Monte Carlo (n_{MC} = 1,000)

important consequences on the computational effort: the CPU time required to solve the problem considering only one eigenvalue is equal to 10s, where the time required without eigenvalue decomposition is equal to 555s. Interestingly, the CPU time with the eigenvalue decomposition but including all eigenvalues is equal to 215s, and thus less than the CPU-time without decomposition. This conclusion depends of course on the dimension of the eigenvalue decomposition, which scales poorly in terms of CPU time.

In order to evaluate the performance of the perturbation algorithm for non-linear stochastic heat transfer (temperature-dependent thermophysical

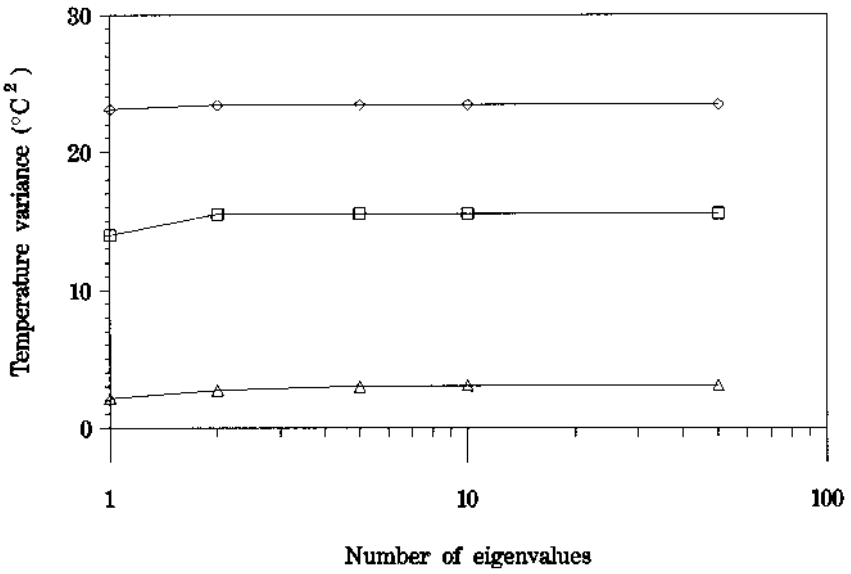

Figure 3.
Influence of the number of eigenvalues in the perturbation algorithm on the accuracy of the variance for x = 0cm and t = 2,000s. ◇ : a/L = 5; □ : a/L = 0.5; Δ : a/L = 0.05

properties), the following temperature dependence of the thermal conductivity and the volumetric heat capacity has been assumed

$$k = k^0(1 + \beta T) \tag{49}$$

$$\rho c = \rho c^0(1 + \beta T) \tag{50}$$

where β is a constant. Further, it has been assumed that k^0 and ρc^0 are Gaussian random variables with the same characteristics as above.

In Figures 4 and 5 the results of the different algorithms are shown. For the computations, k^0 = 100W/m°C, ρc = 7 10^3 J/m^3°C and β = 0.01.

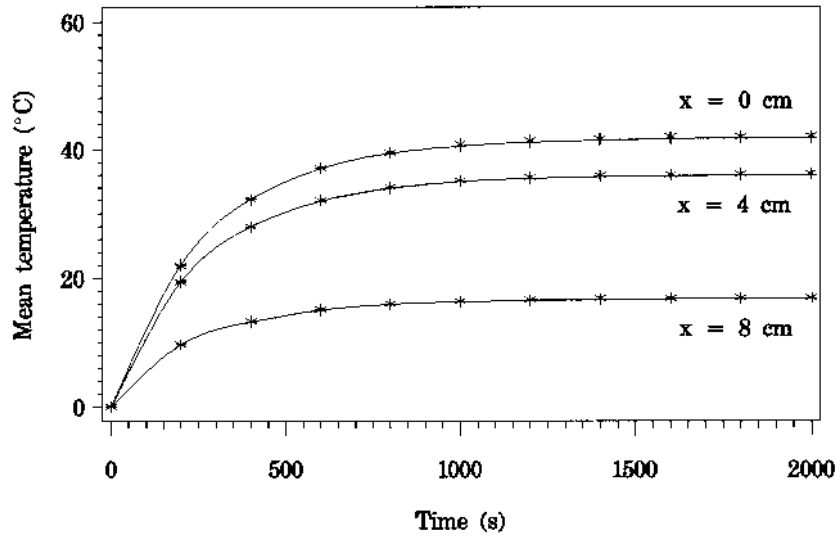

Figure 4.
Mean transient temperature in a slab with random variable non-linear thermophysical properties – perturbation; *: Monte Carlo (n_{MC} = 100); +: Monte Carlo (n_{MC} = 1,000)

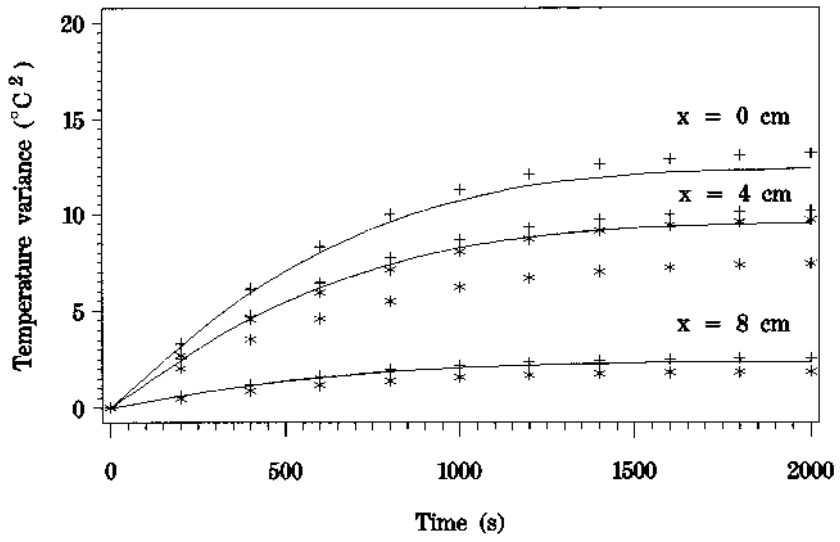

Figure 5.
Variance of the transient temperature in a slab with random variable non-linear thermophysical properties – perturbation; *: Monte Carlo (n_{MC} = 100); +: Monte Carlo (n_{MC} = 1,000)

From the graphs it is clear that the first order perturbation algorithm gives slightly underestimated results. It is, however, still much more accurate than the Monte Carlo method with 100 simulation runs. The difference in CPU time is considerable: 8.7s for the perturbation method, 90.5s for the Monte Carlo method with n_{MC} = 100, and 944s for the Monte Carlo method with n_{MC} = 1,000.

Example 2. Beam with non-square cross-section and internal heat generation

The second test problem consisted of a beam of infinite length with non-square cross-section (Figure 6). A convection boundary condition with $\bar{h}$ = 100W/m^2°C, T_∞ = 0°C, $\sigma_h = 0.1\bar{h}$, σ_{T_∞} = 10°C, and $\sigma_{h,T_\infty} = 0.5\sigma_h\sigma_{T_\infty}$ is applied to all four sides. The statistical characteristics of the thermophysical parameters are equal to $\bar{k}$ =1.0 W/m°C, ρc =1.0 · 10^6 J/m^3°C, $\sigma_k = 0.1\bar{k}$, $\sigma_{\rho c} = 0.1\overline{\rho c}$, $\sigma_{k,\rho c} = 0.5\sigma_k\sigma_{\rho c}$. The beam is initially at a temperature $\bar{T}_0$ = 0°C with σ_{T0} = 10°C. Heat is generated at a rate $\bar{Q}$ = 1 · 10^5 W/m^3°C, with $\sigma_Q = 0.1\bar{Q}$. For all random fields a is set equal to 10cm. The finite element grid consisted of 100 quadrilateral elements with four nodes/elements, yielding a total of 121 nodes. The time step was equal to 60s.

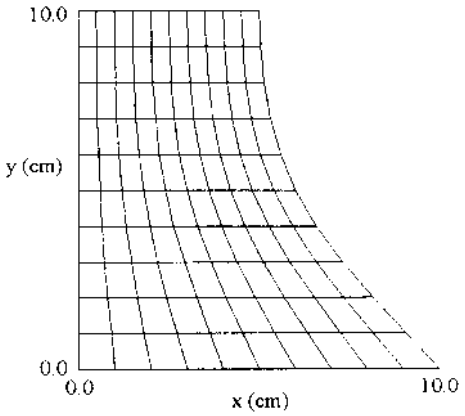

Figure 6.
Finite element grid of a beam with non-square cross-section

In Figures 7 and 8 the mean and the variance of the temperature at position (x, y) = (3cm, 5cm) is shown. Initially the initial temperature fluctuations decrease because of the heat diffusion, but after a while the temperature variance increases again. There is a good correspondence between the results obtained with the different methods. Observe that the results obtained by considering only one eigenvalue agree very well with those obtained by considering all eigenvalues. The CPU time was equal to 648s for the perturbation method without eigenvalue decomposition. For the perturbation method with eigenvalue decomposition the CPU time was equal to 96 and 353s, depending on whether only one or all eigenvalues were considered. The Monte Carlo analysis required 116 (n_{MC} = 100) and 1126s (n_{MC} =1,000). The perturbation method is

hence particularly advantageous when the eigenvalue decomposition of the correlation matrices is accomplished.

Conclusions

In this article a first order perturbation algorithm has been developed for the analysis of non-linear conduction heat transfer under uncertain conditions. The algorithm is based on the finite element formulation of the heat conduction equation, and requires the spatial discretization of the random parameters.

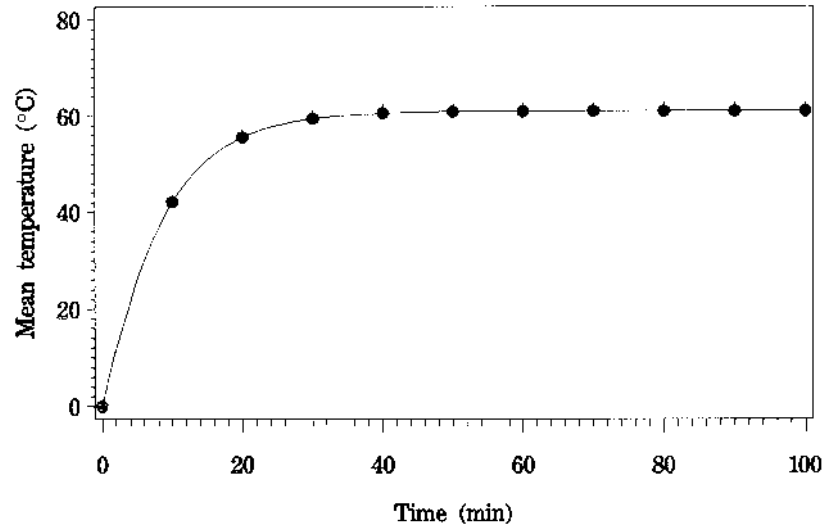

Figure 7.
Mean temperature at the position (x, y) = (3cm, 5cm) of the beam. – perturbation; o: perturbation with one eigenvalue; •: perturbation with all eigenvalues; *: Monte Carlo (n_{MC} = 100); +: Monte Carlo (n_{MC} = 1,000)

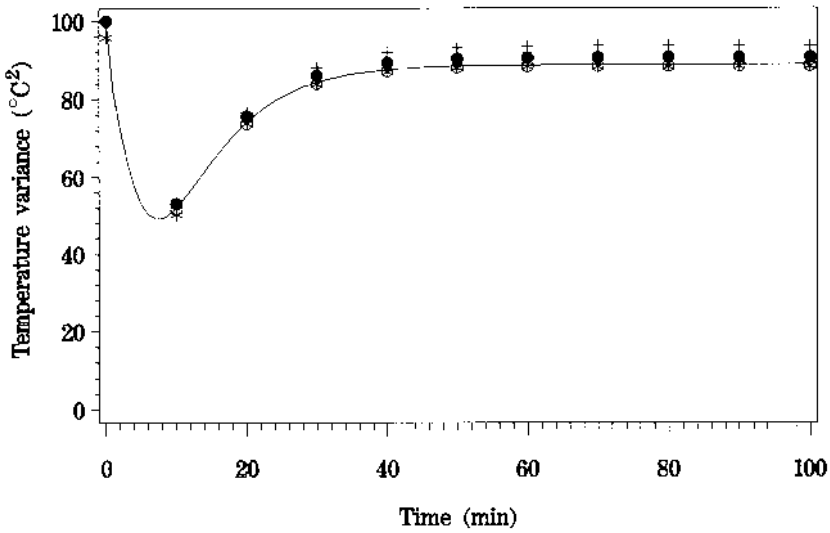

Figure 8.
Temperature variance at the position (x, y) = (3cm, 5cm) of the beam. – perturbation; o: perturbation with one eigenvalue; •: perturbation with all eigenvalues; *: Monte Carlo (n_{MC} = 100; +: Monte Carlo (n_{MC} = 1,000)

In comparison to the perturbation algorithm described by Shuzalec[7] and Fadale and Emery[8], several important features have been added here such as non-linear random field thermophysical parameters, a random field initial condition, and correlated thermophysical and boundary condition parameters. The algorithm is particularly simple to implement, since for each random variable parameter a differential equation of the same type as the the one obtained by applying the finite element method to the deterministic problem must be solved. The perturbation algorithm is a computationally attractive alternative to the Monte Carlo method.

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